ซิมเพิลพัวส์ซองมอดูลที่มีขนาดจำกัดของพีชคณิตพัวส์ซองบางรูป
On finite-dimensional simple Poisson modules of a certain Poisson algebra

Kanokporn Changtong, Ratee Bojaras, Nongkhran Sasom and Sarawut Saenkarun
Department of Mathematics, Statistics and Computer, Faculty of Science,
Ubonratchathani University, Ubonratchathani, Thailand, 34190
*E-mail : kanokporn.c@ubu.ac.th

บทคัดย่อ

เราศึกษาพีชคณิตพัวส์ซอง A ที่มีตัวก่อกำเนิด 3 ตัว ภายใต้ความสัมพันธ์ที่สนใจ และพบว่า A มี พัวส์ซองไอดีลที่ใหญ่ที่สุดทั้งหมด 5 ตัว นอกจากนั้น เราทำการจำแนกซิมเพิลพัวส์ซองมอดูลที่มีมิติจำกัดบน A โดยใช้วิธีที่ปรากฏใน [6] ผลลัพธ์ที่ได้คือ สำหรับแต่ละจำนวนเต็มบวก d จะมีซิมเพิลพัวส์ซองมอดูลที่มีมิติ d อยู่ 5 ตัว

คำสำคัญ : พีชคณิตบน $\mathbb{C}$ พีชคณิตพัวส์ซอง พัวส์ซองไอดีล พัวส์ซองไอดีลที่ใหญ่ที่สุด พัวส์ซองมอดูล

Abstract

A Poisson algebra A with three generators and certain three relations is studied. We show that there are five Poisson maximal ideals of A. Furthermore, we classify the finite-dimensional simple Poisson modules over A by using the method in [6]. The result is that there are five d-dimensional simple Poisson modules for each $d \geq 1$.

Keywords : $\mathbb{C}$-algebra, Poisson algebras, Poisson ideal, Poisson maximal ideal and Poisson modules

Introduction

N. Sasom [7] studied the ring T_q. If q is equal to 1 then T_q becomes the enveloping algebra $U(so_3)$, which is isomorphic to $U(sl_2)$, but it can be viewed as a quantization of the commutative polynomial ring in three determinates. This research classified the finite-dimensional simple T_q-modules using Verma-like modules which are different methods from the work of Havlicek, Klimyk and Posta [4] and [5].

The Poisson algebra is related to T_q. Some authors might call T_q a quantum or quantized algebra, and refer to T_1 as the classical case and Poisson algebra as the semi-classical case. The quantum algebra T_q has five d-dimensional simple modules and the classical case has only one for each dimension d. Now, we ask the same question for the semi-classical case. However, to construct the Poisson algebra, we use the method described in [1, III.5].

The Poisson brackets were introduced by Joseph-Louis Lagrange and his student, Siméon Denis Poisson, at the beginning of 19th century, as an algorithm useful to produce solutions of the equations of motion. In the seventies of 19th century Marius Sophus Lie began his deep researches on the geometry of P.D.E. The more systematic study of Poisson brackets was started, and Lie described new examples of Poisson brackets, which are different from the sense presented by the Poisson and Lagrange's ones. These brackets are now called Lie-Poisson brackets and were ignored by mathematicians, until Kirilov, Konstant and Souriau redefined them in the context of the theory of Lie groups representations and geometric quantization theory. However, A Poisson algebra is named in honour of Siméon Denis Poisson.

A Poisson algebra is a commutative $\mathbb{C}$-algebra equipped with a Lie bracket $\{-,-\}$ for which $\{a,-\}: A \longrightarrow A$ is always a derivation of A. The bracket $\{-,-\}$ is then called a *Poisson bracket*. A *Poisson ideal* is an ideal of A for both the algebra structures on A.

If T is a $\mathbb{C}$-algebra with a central non-unit non-zero-divisor t such that $A := T/tT$ is commutative then there is a Poisson bracket $\{-,-\}$ on A such that $\{\bar{x},\bar{y}\} = \overline{t^{-1}[x,y]}$ for all $\bar{x},\bar{y} \in A$.

N. Sasom [7] considered Poisson A-modules in the aspect of [2] and classified the finite-dimensional simple Poisson modules for the Poisson algebra $A = \mathbb{C}[x,y,z]$ with Poisson bracket

$$\{x,y\} = yx + z,$$
$$\{y,z\} = zy + x, \text{ and}$$
$$\{z,x\} = xz + y\,.$$

The aim of this research is to classify the finite-dimensional simple Poisson modules for the Poisson algebra with the other certain Poisson bracket, namely, the Poisson algebra $A = \mathbb{C}[x,y,z]$ with Poisson bracket :

$$\{x,y\} = yx + z - x - y,$$
$$\{y,z\} = zy + x - y - z, \text{ and}$$
$$\{z,x\} = xz + y - x - z.$$

By the routine calculation we can see that there are five Poisson maximal ideals for this Poisson algebra, and we use the same method described in [6] to classify its finite-dimensional simple Poisson modules.

Definitions and Notations

Throughout this paper, A will be a finitely generated commutative algebra over $\mathbb{C}$.

Definition 2.1. A *Poisson bracket* on A is a Lie algebra bracket $\{-,-\}$ satisfying the Leibniz rule $\{\mathrm{ab},\mathrm{c}\} = \mathrm{a}\{\mathrm{b},\mathrm{c}\} + \{\mathrm{a},\mathrm{c}\}\mathrm{b}$ for all $\mathrm{a},\mathrm{b},\mathrm{c} \in \mathrm{A}$. The pair $(\mathrm{A},\{-,-\})$ is called a *Poisson algebra*.

A subalgebra B of A is a *Poisson subalgebra* of A if $\{b,c\} \in B$ for all $b,c \in B$ and an ideal I of A is a *Poisson ideal* if $\{i,a\} \in I$ for all $i \in I$ and all $a \in A$. If I is a Poisson ideal of A then A/I is a Poisson algebra where $\{a + I, b + I\} = \{a,b\} + I$. A Poisson algebra A is said to be *simple* if its only Poisson ideals are 0 and A.

Definition 2.2. Let P be an ideal of a Poisson algebra A. Then P is a *Poisson prime ideal* if P is both a prime ideal and a Poisson ideal. It follows from [5] that this is equivalent to saying that P is a Poisson ideal and, for all Poisson ideals $\mathrm{I},\mathrm{J} \subseteq \mathrm{A}$, $\mathrm{I}\,\mathrm{J} \subseteq \mathrm{P}$ implies that $\mathrm{I} \subseteq \mathrm{P}$ or $\mathrm{J} \subseteq \mathrm{P}$.

By *maximal Poisson ideal*, we mean a Poisson ideal I of A such that if J is a Poisson ideal and $I \nsubseteq J$ then $J = A$. By *Poisson maximal ideal*, we mean a maximal ideal of A that is also a Poisson ideal. For example, let $A = \mathbb{C}[x,y]$ which is a Poisson algebra with the Poisson bracket $\{x,y\} = 1$.Then 0 is a maximal Poisson ideal but is not a Poisson maximal ideal.

Let R be a commutative $\mathbb{C}$-algebra and let $h \in R$. Let A be an R-algebra and suppose that h is not a zero divisor in A, and that $\bar{A} = A/hA$ is a commutative $\mathbb{C}$-algebra. Then there is a Poisson bracket $\{-,-\}$ on $\bar{A}$ such that $\{\bar{a},\bar{b}\} = \overline{h^{-1}[a,b]}$ for all $\bar{a} = a + hA$ and $\bar{b} = b + hA$. Following [1], we call A a quantization of the Poisson algebra $\bar{A}$.

Note that there are more than one definition of Poisson module the literature. We shall use the one introduced by D. R. Farkas [3].

Definition 2.3. Let A be a commutative Poisson algebra with Poisson bracket $\{-,-\}$. We shall say that an A-module M is a *Poisson module* if there is a bilinear form $\{-,-\}_M : A \times M \to M$ such that

(i) $\{a, a'm\}_M = \{a,a'\}m + a'\{a,m\}_M$ for all $a, a' \in A$ and all $m \in M$.

(ii) $\{aa', m\}_M = a\{a',m\}_M + a'\{a,m\}_M$ for all $a, a' \in A$ and all $m \in M$.

(iii) $\{\{a,a'\},m\}_M = \{a,\{a',m\}\}_M - \{a',\{a,m\}_M\}_M$ for all $a, a' \in A$ and all $m \in M$.

A submodule N of a Poisson module M is called a *Poisson submodule* if $\{a,n\}_M \in N$ for all $a \in A, n \in N$.

Definition 2.4. Let N be a left module over a ring R. Give any subset $X \subseteq N$, the *annihilator* of X is the set

$ann_R(X) = \{r \in R : rx = 0 \quad \text{for all} \quad x \in X\}$,

which is a left ideal of R.

Lemma 2.5. Let A be a Poisson $\mathbb{C}$-algebra and M a Poisson A-module.

(i) The annihilator of M is a Poisson ideal of A;

(ii) if M is a simple Poisson module then the annihilator of M is a prime Poisson ideal of A;

(iii) if M is a finite-dimensional simple Poisson module then the annihilator of M is a Poisson maximal ideal of A.

Proof. See [7].

The following lemma shows that instead of checking the structure of a Poisson module over a polynomial Poisson algebra, we only have to check the axioms on the generators.

Lemma 2.6. Let $A = \mathbb{C}[x_1, x_2, \ldots, x_n]$ with a Poisson bracket $\{-,-\}$. Let $V = Sp(x_1, x_2, \ldots, x_n)$ and let M be an A-module. Suppose that there is a bilinear form $\{-,-\}_M : V \times M \to M$. Extend this to a bilinear form $\{-,-\}_M : A \times M \to M$ using Definition 2.3(ii) and $\{1,m\}_M = 0$. If Definition 2.3(i) and (iii) hold, for all $m \in M$, whenever $a = x_i$ and $a' = x_j$ for $1 \le i < j \le n$ then Definition 2.3(i) and (iii) hold for all $a, a' \in A$.

Proof. See [7].

Let $(R\{-,-\})$ be a Poisson algebra. We say that a $\mathbb{C}$-algebra automorphism $\theta: R \to R$ is a *Poisson automorphism* if for all $x, y \in R, \theta(\{x,y\}) = \{\theta(x),\theta(y)\}$. Poisson automorphisms can be used to twist the module structures of a Poisson modules, as specified in the following theorem.

Theorem 2.7. Let R be a commutative Poisson algebra with a $\mathbb{C}$-algebra automorphism α. Let $r,s,t \in R$. If $\{\alpha(r),\alpha(s)\} = \alpha\{r,s\}$ and $\{\alpha(r),\alpha(t)\} = \alpha\{r,t\}$ then $\{\alpha(r),\alpha(s+t)\} = \alpha\{r,s+t\}$ and $\{\alpha(r),\alpha(st)\} = \alpha\{r,st\}$. Therefore if X is a set of generators of R and

$$(\alpha\{x,y\}) = \{\alpha(x),\alpha(y)\}$$

for all $x, y \in X$ then α is a Poisson automorphism of R.

Proof. See [6].

Theorem 2.8. Let A be a Poisson algebra, let α be a Poisson automorphism of A and let M be a Poisson module. Define $a.m = \alpha(a)m$

and $\{a,m\}_M^\alpha = \{\alpha(a), m\}_M$ for all $a \in A$ and $m \in M$. Then M is a Poisson module under $-.-: A \times M \to M$ and $\{-,-\}_M^\alpha: A \times M \to M$.

Proof. See [7].

Let M be a module.

(i) We denote the module M constructed in Theorem 2.8 by M^α.

(ii) The annihilator of M^α, $ann_A M^\alpha = \alpha^{-1}(ann_A M)$.

(iii) The Poisson submodules of M^α have the form N^α where N is a Poisson submodule of M.

(iv) A module M^α is a simple Poisson module if and only if M is a simple Poisson module.

Results and Discussion

Let T be the $\mathbb{C}$-algebra generated by x, y, z, t and t^{-1} subject to the relations

$$xy - tyx = (t-1)(z-x-y),$$
$$yz - tzy = (t-1)(x-y-z),$$
$$zx - txz = (t-1)(y-x-z), \text{ and}$$
$$xt = tx, \quad yt = ty, \quad zt = tz, \quad tt^{-1} = 1 = t^{-1}t.$$

Then t is a central element of T.

Let

$$A := T/(t-1)T \simeq \mathbb{C}[x,y,z],$$

which is a commutative polynomial algebra. The induced Poisson bracket on A is such that

$$\{x,y\} = \frac{1}{t-1}[x,y]$$
$$= \frac{1}{t-1}(xy - yx)$$
$$= \frac{1}{t-1}(tyx + (t-1)(z-x-y) - yx)$$
$$= yx + z - x - y.$$

Here we are abusing notation by writing x, y and z for both an element of T and its image in A. Similarly, we obtain

$\{y,z\} = zy + x - y - z,\ \{z,x\} = xz + y - x - z.$

In the next lemma, we find the Poisson maximal ideals of A for this Poisson bracket.

Lemma 3.1. In the above Poisson algebra A, there are only five Poisson maximal ideals of A which are the followings:

$$J_1 = (x-1)A + (y-1)A + (z-1)A,$$
$$J_2 = xA + yA + zA,$$
$$J_3 = xA + (y-2)A + (z-2)A,$$
$$J_4 = (x-2)A + yA + (z-2)A, \text{ and}$$
$$J_5 = (x-2)A + (y-2)A + zA.$$

Proof. Let J be a Poisson maximal ideal of A. Since A is a commutative polynomial ring over $\mathbb{C}$,

$J = (x-a, y-b, z-c)$ for suitable $a, b, c \in \mathbb{C}$. As J is Poisson, $\{x, J\} \subseteq J$, $\{y, J\} \subseteq J$, and $\{z, J\} \subseteq J$. Observe that

$$yx + z - x - y = \{x, y-b\} \in J,$$
$$-(zx + y - x - z) = \{x, z-c\} \in J,$$
$$zy + x - y - z = \{y, z-c\} \in J.$$

This happens precisely when $ab + c - a - b = ac + b - a - c = bc + a - b - c = 0$. As $ab + c - a - b = ac + b - a - c$, we have $ab + ac - 2a = 0$. This implies that $a = 0$ or $b = 2 - c$. Similarly $c = 0$ or $b = 2 - a$ and $b = 0$ or $a = 2 - c$. If $a = 0$ then $a = b = c = 0$ or $a = 0, b = 2, c = 2$ and if $b = 2 - c$ then $a = 2, b = 2, c = 0$ or $a = 2, b = 0, c = 2$ or $a = 1, b = 1, c = 1$. Similarly for $c = 0$ or $b = 2 - a$ and $b = 0$ or $a = 2 - c$. So there are five solutions:

(i) $a = b = c = 0$;

(ii) $a = b = c = 1$;

(iii) $a = b = 2$ and $c = 0$;

(iv) $b = c = 2$ and $a = 0$;

(v) $a = c = 2$ and $b = 0$.

We will classify finite-dimensional simple Poisson A-modules annihilated by J_1 as follows.

Lemma 3.2. Let M be a Poisson module annihilated by $J_1 = (x-1)A + (y-1)A + (z-1)A$ and let $m \in M$.
Then we have :

(i) $xm = ym = zm = m$.

(ii) (a) $\{xy, m\}_M = \{y, m\}_M + \{x, m\}_M$;

(b) $\{yz, m\}_M = \{y, m\}_M + \{z, m\}_M$;

(c) $\{xz, m\}_M = \{z, m\}_M + \{x, m\}_M$.

(iii) (a) $\{z, m\}_M = \{x, \{y, m\}_M\}_M - \{y, \{x, m\}_M\}_M$;

(b) $\{x, m\}_M = \{y, \{z, m\}_M\}_M - \{z, \{y, m\}_M\}_M$;

(c) $\{y, m\}_M = \{z, \{x, m\}_M\}_M - \{x, \{z, m\}_M\}_M$.

Proof. This is a routine calculation.

Let M be a Poisson module annihilated by J. Let $m \in M$ be an eigenvector for $\{x, -\}_M$ with eigenvalue $\lambda \in \mathbb{C}$. Thus $\{x, m\}_M = \lambda m$. It follows from Lemma 3.2 (iii) that

(i) $\{x, \{y, m\}_M\}_M = \{z, m\}_M + \lambda\{y, m\}_M$;

(ii) $\{x, \{z, m\}_M\}_M = \lambda\{z, m\}_M - \{y, m\}_M$;

(iii) $\{y, \{z, m\}_M\}_M - \{z, \{y, m\}_M\}_M = \{x, m\}_M = \lambda m$.

To simplify these, we shall replace y and z by $u := \frac{1}{2}(y - iz)$ and $v := \frac{1}{2}(z - iy)$, respectively.

Lemma 3.3. Let $A = \mathbb{C}[x, y, z]$ be equipped with the Poisson bracket

$$\{x, y\} = yx + z - x - y,$$
$$\{y, z\} = zy + x - y - z, \text{ and}$$
$$\{z, x\} = xz + y - x - z.$$

If $u := \frac{1}{2}(y - iz)$ and $v := \frac{1}{2}(z - iy)$, then u, v, x generate A and the Poisson bracket is given by

$$\{x, v\} = -ixu - iv + iu + \tfrac{1}{2}x(i+1),$$
$$\{x, u\} = ixv + iu - iv - \tfrac{1}{2}x(i+1),$$
$$\{u, v\} = \tfrac{1}{2}(x - u - v + i(u^2 + v^2 - u - v)).$$

Proof. This is a routine calculation.

Lemma 3.4. Let M be a Poisson module annihilated by J_1 and let $m \in M$. Then we have;

(i) $xm = m,\ um = vm = \frac{1}{2}(1 - i)m$;

(ii) $\{xu, m\}_M = \frac{1}{2}(1 - i)\{x, m\}_M + \{u, m\}_M$,
$\{xv, m\}_M = \frac{1}{2}(1 - i)\{x, m\}_M + \{v, m\}_M$,
$\{uv, m\}_M = \frac{1}{2}(1 - i)\{u, m\}_M + \frac{1}{2}(1 - i)\{v, m\}_M$,
$\{u^2, m\}_M = (1 - i)\{u, m\}_M$,
$\{v^2, m\}_M = (1 - i)\{v, m\}_M$.

(iii) (a) $\{x, \{u, m\}_M\}_M - \{u, \{x, m\}_M\}_M = i\{u, m\}_M$;

(b) $\{x, \{v, m\}_M\}_M - \{v, \{x, m\}_M\}_M = -i\{v, m\}_M$;

(c) $\{u\{v, m\}_M\}_M - \{v, \{u, m\}_M\}_M = \frac{1}{2}i\{x, m\}_M$.

Proof. This is a routine calculation.

Lemma 3.5. Let $u := \frac{1}{2}(y - iz)$ and $v := \frac{1}{2}(z - iy)$ (so that $y = u + iv$ and $z = v + iu$). Let M be a Poisson module annihilated by $J := J_1$. Let $\lambda \in \mathbb{C}$ be such that $\{x, m\}_M = \lambda m$ for some $0 \neq m \in M$. Then

(i) $\{x, \{v, m\}_M\}_M = (\lambda - i)\{v, m\}_M$,

(ii) $\{x, \{u, m\}_M\}_M = (\lambda + i)\{u, m\}_M$,

(iii) $\{u, \{v, m\}_M\}_M - \{v, \{u, m\}_M\}_M = \frac{1}{2}\lambda m$.

Proof. This is immediate from Lemma 3.4.

Lemma 3.6. Let $\mathbb{C}[x, u, v]$ with the Poisson bracket as in Lemma 3.3. Let $d \geq 1$. There is a d-dimensional Poisson A-module M, with basis $\{m_1, m_2, \dots, m_d\}$, such that $xm = m, um = vm = \frac{1}{2}(1 - i)$ and

(i) $\{x, m_j\}_M = (\lambda + (j-1)i)m_j$ for $1 \leq j \leq d$,

(ii) $\{v, m_1\}_M = 0$ and
$\{v, m_j\}_M = -\frac{1}{2}(j-1)(\lambda + \frac{1}{2}(j-2)i)m_{j-1}$ for $1 < j \leq d$,

(iii) $\{u, m_j\}_M = m_{j+1}$ for $1 \leq j < d$ and

$\{u, m_d\}_M = 0$,
where $\lambda = \frac{1-d}{2} i$.

Proof. By Lemma 2.6, it is enough to show that Definition 2.3 (i) and (iii) hold for $m = m_j$ and $(a, a') = (x, u), (x, v)$ or (u, v) for the brackets defined (i), (ii) and (iii) above. We then extend
the Poisson action on M from $V := \mathbb{C}x + \mathbb{C}u + \mathbb{C}v$ to $\mathbb{C}[x, u, v]$ using Definition 2.3 (ii). Then the conclusions of Lemma 3.4 (i) and (ii) hold and M becomes a Poisson module with the stated properties.

Lemma 3.7. Let $d \geq 1$. The d-dimensional Poisson module constructed in Lemma 3.6 is simple as a Poisson module.

Proof. Let $\lambda_j = \lambda + (j-1)i$, $1 \leq j \leq d$. Note that $\lambda_j \neq \lambda_k$ when $j \neq k$. Let N be a non-zero Poisson submodule of M. Let $0 \neq n = \sum_{j=1}^{d} \alpha_j m_j \in N$ be such that minimally many of the coefficients $\alpha_j \in \mathbb{F}$ are non-zero and choose k so that $\alpha_k \neq 0$.

$$\{x, n\}_M - \lambda_k n = \sum_{j=1}^{d} \alpha_j (\lambda_j - \lambda_k) m_j.$$

This has one fewer non-zero coefficient than n so, by minimality, it is 0 and hence $\alpha_j = 0$ when $j \neq k$, that is $n = \alpha_k m_k$. Therefore $m_k \in N$. By the Poisson action of u and v, $m_j \in N$ for all j. So $N = M$ and M is a simple Poisson module.

We now aim to show that the simple d-dimensional Poisson module constructed above is unique.

Lemma 3.8. Let M be a finite-dimensional simple Poisson module annihilated by

$$J_1 = xA + (u - \tfrac{1}{2}(1-i))A + (v - \tfrac{1}{2}(1-i))A,$$

and let $n \leq \dim M$. There exist $\lambda \in \mathbb{C}$ and n linearly independent elements $m_1, m_2, \ldots, m_n \in M$ such that

(i) $\{x, m_j\}_M = (\lambda + (j-1)i) m_j$ for $1 \leq j \leq n$
(ii) $\{v, m_1\}_M = 0$ and
$\{v, m_j\}_M = -\frac{1}{2}(j-1)(\lambda + \frac{1}{2}(j-2)i) m_{j-1}$,
for $1 < j \leq n$
(iii) $\{u, m_j\}_M = m_{j+1}$, for $1 \leq j < n$.

Proof. Let $\Lambda = \{\lambda \in \mathbb{C} : \{x, m\}_M = \lambda m$ for some $0 \neq m \in M\}$. Since $\dim_{\mathbb{C}} M < \infty$, the linear transformation of M, with $m \mapsto \{x, m\}_M$, has an eigenvalue, therefore $\Lambda \neq \emptyset$. As x, u, v generate A, it follows from Lemma 3.4 (iii) and Lemma 3.5 (ii) that if $\lambda \in \Lambda$ then $\{m \in M : \{x, m\}_M = (\lambda + ni) m$ for some $n \in \mathbb{Z}\}$ spans a non-zero Poisson module of M. As M is finite-dimensional, $\lambda \in \Lambda$ can be chosen so that $\lambda - i \notin \Lambda$. Let m_1 be an eigenvector for $\{x, -\}_M$ with eigenvalue λ. By Lemma 3.5 (ii), $\{x, \{v, m_1\}_M\}_M = (\lambda - i)\{v, m_1\}_M$ so $\{v, m_1\}_M = 0$. Thus the result is true when $n = 1$. We proceed by induction on n. Suppose that (i), (ii) and (iii) hold for n and that $n + 1 \leq \dim M$. Then $\{u, m_n\}_M \neq 0$, otherwise $Sp\,(m_1, m_2, \ldots, m_n)$ is an
n-dimensional Poisson submodule of M, contrary to the Poisson simplicity of M. Let $m_{n+1} = \{u, m_n\}_M$. By Lemma 3.5 (ii) $\{x, m_n\}_M = (\lambda + (n-1)i) m_n$ and $\{x, m_{n+1}\}_M = \{x, \{u, n\}_M\}_M = (\lambda + ni) m_{n+1}$, in accordance with (iii) for $j = n + 1$. By Lemma 3.5 (iii),
$\{u, \{v, m_n\}_M\}_M - \{v, \{u, m_n\}_M\}_M = \frac{1}{2}\{x, m_n\}_M$.
So

$$-\frac{1}{2}(n-1)\left(\lambda + \frac{1}{2}(n-2)i\right)\{u, m_{n-1}\}_M - \{v, m_{n+1}\}_M = \frac{1}{2}(\lambda + (n-1)i) m_n$$

and hence,

$$\{v, m_{n+1}\}_M = -\tfrac{1}{2}(n-1)\left(\lambda + \tfrac{1}{2}(n-2)i\right)m_n - \tfrac{1}{2}(\lambda + (n-1)i)m_n.$$

It follows that

$$\{v, m_{n+1}\}_M = -\frac{1}{2}n\left(\lambda + \frac{1}{2}(n-1)i\right)m_n.$$

Note that, being eigenvectors for $\{x, -\}_M$ with distinct eigenvalues, $m_1, m_2, \ldots, m_{n+1}$ are linearly independent. The result holds by induction on n.

Theorem 3.9. Let M be a finite-dimensional simple Poisson module annihilated by

$$J_1 = xA + (u - \tfrac{1}{2}(1-i))A + (v - \tfrac{1}{2}(1-i))A,$$

and let $d = \dim_{\mathbb{C}} M$. There exist d linearly independent elements $m_1, m_2, \ldots, m_d \in M$ such that

(i) $\{x, m_j\}_M = (\lambda + (j-1)i)m_j$, for $1 \le j \le$d;
(ii) $\{v, m_j\}_M = -\frac{1}{2}(j-1)(\lambda + \frac{1}{2}(j-2)i)m_{j-1}$, for $1 \le j \le$d;
(iii) $\{u, m_j\}_M = m_{j+1}$, for $1 \le j <$d, and $\{u, m_d\}_M = 0$,

where $= \frac{1-d}{2}i$.

Proof. By Lemma 3.8, there exist $\lambda \in \mathbb{C}$ and linearly independent elements $m_1, m_2, \ldots, m_d$ of M such that

(i) $\{x, m_j\}_M = (\lambda + (j-1)i)m_j$, for $1 \le j \le$d;
(ii) $\{v, m_j\}_M = -\frac{1}{2}(j-1)(\lambda + \frac{1}{2}(j-2)i)m_{j-1}$, for $1 \le j \le$d;
(iii) $\{u, m_j\}_M = m_{j+1}$, for $1 \le j <$d.

As $\dim_{\mathbb{C}} M = d$, $M = Sp(m_1, m_2, \ldots, m_d)$ and is the sum of the eigenspaces for $\{x, -\}_M$and the eigenvalues $\lambda, \lambda + i, \ldots, \lambda + (d-1)i$. By Lemma 3.5,

$$\{x, \{u, m_d\}_M\}_M = (\lambda + di)\{u, m_d\}_M$$

but $\lambda + di$ is not an eigenvalue of $\{u, m_d\}_M$ for $\{x, -\}_M$ so $\{u, m_d\}_M = 0$. By Lemma 3.5 (iii) $\{u, \{v, m_d\}_M\}_M - \{v, \{u, m_d\}_M\}_M = \frac{1}{2}\{x, m_d\}_M$. We obtain

$$-\frac{1}{2}\left(\lambda + \frac{1}{2}(d-2)i\right)\{u, m_{d-1}\}_M = \tfrac{1}{2}(\lambda + (d-1)i)m_d.$$

That is

$$-\tfrac{1}{2}\left(\lambda + \tfrac{1}{2}(d-2)i\right)m_d = \tfrac{1}{2}(\lambda + (d-1)i)m_d,$$

so

$$d(\lambda + \frac{1}{2}(d-1)i) = 0$$

from which it follows that $\lambda = \frac{1-d}{2}i$.

We conclude from Lemma 3.6 and Theorem 3.9 that for $d \ge 1$ there is a unique d-dimensional simple Poisson module M annihilated by J_1 as the following theorem.

Theorem 3.10. Let $d \ge 1$. There is a unique d-dimensional simple Poisson module over A, annihilated by J_1. It has a basis $m_1, m_2, \ldots, m_d$ such that

(i) $\{x, m_j\}_M = (\lambda + (j-1)i)m_j$ for $1 \le j \le$ d;
(ii) $\{y, m_1\}_M = m_2$,
$\{y, m_j\}_M = m_{j+1} - \frac{1}{2}i(j-1)(\lambda + \frac{1}{2}(j-2))m_{j-1}$ for $1 < j <$ d and
$\{y, m_d\}_M = -\frac{1}{2}i(d-1)(\lambda + \frac{1}{2}(d-2)i)m_{d-1}$;
(iii) $\{z, m_1\}_M = im_2$,
$\{z, m_j\}_M = im_{j+1} - \frac{1}{2}(j-1)(\lambda + \frac{1}{2}(j-2)i)m_{j-1}$,
for $1 < j <$d and
$\{z, m_d\}_M = -\frac{1}{2}(d-1)(\lambda + \frac{1}{2}(d-2)i)m_{d-1}$,
where $\lambda = \frac{1-d}{2}i$.

Proof. This is immediate from Lemma 3.3, Lemma 3.6 and Theorem 3.9**.**

We now consider Poisson modules annihilated by $J_2 = xA + yA + zA$.

Lemma 3.11. Let M be a Poisson module annihilated by

$$J_2 = xA + yA + zA.$$

and let $m \in M$. Then we have :

(i) $xm = ym = zm = -m.$

(ii) $\{yx, m\}_M = \{zy, m\}_M = \{xz, m\}_M = 0$

(iii) (a) $\{x, \{y, m\}_M\}_M - \{y, \{x, m\}_M\}_M = \{z, m\}_M - \{x, m\}_M - \{y, m\}_M;$

(b) $\{y, \{z, m\}_M\}_M - \{z, \{y, m\}_M\}_M = \{x, m\}_M - \{y, m\}_M - \{z, m\}_M;$

(c) $\{z, \{x, m\}_M\}_M - \{x, \{z, m\}_M\}_M = \{y, m\}_M - \{x, m\}_M - \{z, m\}_M.$

Proof. This is a routine calculation.

Let M be a Poisson module annihilated by J_2 and let $m \in M$ be an eigenvector for $\{x, -\}_M$ with eigenvalue $\lambda \in \mathbb{C}$. Then $\{x, m\}_M = \lambda m$. It follows from Lemma 3.11 (iii) that

(i) $\{x, \{y, m\}_M\}_M = (\lambda - 1)\{y, m\}_M + \{z, m\}_M - \lambda$m;

(ii) $\{x, \{z, m\}_M\}_M = -\{y, m\}_M + \lambda\mathrm{m} + \{x, m\}_M + (\lambda + 1)\{z, m\}_M;$

(iii) $\{y, \{z, m\}_M\}_M - \{z, \{y, m\}_M\}_M = \lambda\mathrm{m} - \{y, m\}_M - \{z, m\}_M$

The method used to classify finite-dimensional simple Poisson module annihilated by J_1 can be applied to classify those annihilated by J_2. Here we change generators to x, y and $u := z - x - y$ so that $J_2 = xA + yA + uA$ and the Poisson bracket is:

$$\{x, y\} = xy + u,$$
$$\{x, u\} = x(2 - x - 2y - u),$$
$$\{y, u\} = y(u + 2x + y - 2).$$

Adapting the proofs of Lemmas 4.2, 4.4, 4.5 and 4.6 and Theorems 4.7 and 4.8, we obtain the following theorem.

Theorem 3.12. Let $d \geq 1$. There is a unique d-dimensional simple Poisson module over A, annihilated by J_2. It has a basis $m_1, m_2, \dots, m_d$ such that

(i) $\{x, m_j\}_M = (j-1)(\lambda + j - 2)m_{j-1}$, for $1 \leq j \leq d$;

(ii) $\{y, m_j\}_M = m_{j+1}$ for $1 \leq j <$d, and $\{y, m_d\}_M =0$,

(iii) $\{z, m_j\}_M = (j-1)(\lambda + j - 2)m_{j-1} + (\lambda + 2(j-1))m_j + m_{j+1}$, for $1 \leq j <$d,

And

$$\{z, m_d\}_M = (d-1)(\lambda + d - 2)m_{d-1} + (\lambda + 2(d-1))m_d,$$

where $\lambda = 1 - d$.

Similarly to J_1 and J_2 by replacing x by $u = x + y + z$, we obtain the following theorem.

Theorem 3.13. Let $d \geq 1$. There is a unique d-dimensional simple Poisson module over A, annihilated by J_3. It has a basis $m_1, m_2, \dots, m_d$ such that

(i) $\{x, m_j\}_M = (\lambda + 2(j-1))m_j - (j-1)(\lambda + j - 2)m_{j-1} - m_{j+1}$, for $1 \leq j < d$;

and

$$\{x, m_d\}_M = (\lambda + 2(d-1))m_d - (d-1)(\lambda + d - 2)m_{d-1},$$

(ii) $\{y, m_j\}_M = (j-1)(\lambda + (j-2))m_{j-1}$, for $1 \leq j \leq d$,

(iii) $\{z, m_j\}_M = m_{j+1}$, for $1 \leq j <$d and $\{z, m_d\}_M = 0$,

where $\lambda = 1 - d$.

To classify the general forms of finite-dimensional simple Poisson modules annihilated by J_j where $j = 4, 5$ we make use of Poisson automorphisms A.The Poisson bracket of A is

$$\{x,y\} = yx + z - x - y,$$
$$\{y,z\} = zy + x - y - z,$$
$$\{z,x\} = xz + y - x - z$$

Let α, β and γ be the $\mathbb{C}$-automorphisms of A such that

(i) $\alpha(x) = y, \alpha(y) = x, \alpha(z) = z,$
(ii) $\beta(x) = z, \beta(y) = y, \beta(z) = x.$

Then we can check, using Theorem 2.7, that α and β are Poisson automorphisms of A. Observe that $\alpha(J_3) = J_4$ and $\beta\ (J_3) = J_5$. As $\alpha^2 = \beta^2 = id$, the simple Poisson modules annihilated by J_4 and J_5 are precisely the Poisson modules M^α and M^β where M is a simple Poisson module annihilated by J_3. We can conclude that

for each $d \geq 1$ there is precisely one d-dimensional simple Poisson module annihilated by J_4 and J_5 respectively. By combining the above results, we have the theorem.

Theorem 3.14. For $d \geq 1$, the Poisson algebra A has precisely five d-dimensional simple Poisson modules.

Proof. This is immediate from Theorem 3.10., Theorem 3.12. and Theorem 3.13. using the Poisson automorphisms of A.

Conclusion

For a Poisson algebra A with three generators and certain three relations, there are five Poisson maximal ideals and there are five d-dimensional simple Poisson modules for each $d \geq 1$. As it is seen in this study. the method in this research is based on the calculation which makes it quite complicated. The further work is to study the n ew method and find out how it can be applied to this Poisson algebra A. This new method is interested in the derived algebra of $J/_{J^2}$ where J is a Poisson maximal ideal of A.

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